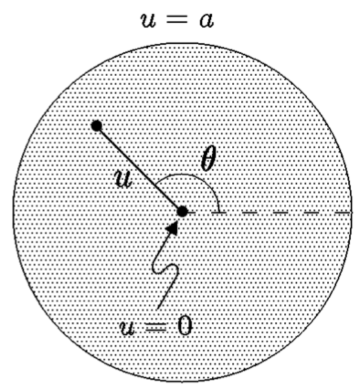
## PROBLEM 18: A TWO-DIMENSIONAL CURVED SPACE *(40 points)*

*The following problem was Problem 3, Quiz 2, 2002.*

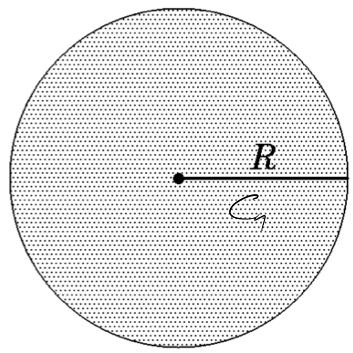
Consider a two-dimensional curved space described by polar coordinates $u$ and $\theta$, where $0 \le u \le a$ and $0 \le \theta \le 2\pi$, and $\theta = 2\pi$ is as usual identified with $\theta = 0$. The metric is given by




$$ds^2 = \frac{a\,du^2}{4u(a-u)} + u\,d\theta^2 .$$

A diagram of the space is shown at the right, but you should of course keep in mind that the diagram does not accurately reflect the distances defined by the metric.

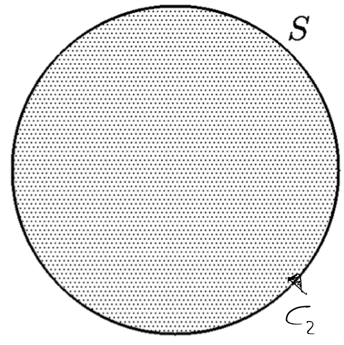
(a) *(6 points)* Find the radius $R$ of the space, defined as the length of a radial (i.e., $\theta =$ *constant*) line. You may express your answer as a definite integral, which you need not evaluate. Be sure, however, to specify the limits of integration.




$$ds^2\Big|_{d\theta=0} = \frac{a\,du^2}{4u(a-u)} \Rightarrow ds\Big|_{d\theta=0} = \sqrt{\frac{a}{4u(a-u)}}\,du$$

$$\Rightarrow \quad R = \int_{C_1} ds\Big|_{d\theta=0} = \int_0^a \sqrt{\frac{a}{4u(a-u)}}\,du$$

(b) *(6 points)* Find the circumference $S$ of the space, defined as the length of the boundary of the space at $u = a$.

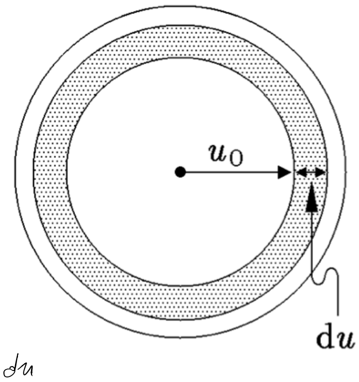



$$ds^2\Big|_{du=0} = u\,d\theta^2 \Rightarrow ds\Big|_{C_2} = \sqrt{a}\,d\theta$$

$u = a$ on $C_2$

$$\Rightarrow \quad S = \int_{C_2} ds = \int_0^{2\pi} \sqrt{a}\,d\theta = 2\pi\sqrt{a}$$

(c) *(7 points)* Consider an annular region as shown, consisting of all points with a $u$-coordinate in the range $u_0 \le u \le u_0 + \mathrm{d}u$. Find the physical area $\mathrm{d}A$ of this region, to first order in $\mathrm{d}u$.




(line element $u$) (line element $\theta$)

$$dA = \sqrt{\frac{a}{4u(a-u)}}\, du \cdot \sqrt{u}\, d\theta$$

$$\Longrightarrow \quad dA\Big|_{\text{annulus}} = \int_{u_0}^{u_0+du} du' \int_0^{2\pi} d\theta \sqrt{\frac{a}{4u'(a-u')}} \sqrt{u'} = \frac{2\pi}{2} \int_{u_0}^{u_0+du} \sqrt{\frac{a}{a-u'}}\, du'$$

$$\approx \pi \sqrt{\frac{a}{a-u_0}}\, du \quad //$$

(d) *(3 points)* Using your answer to part (c), write an expression for the total area of the space.

$$A = \int_0^a \pi \sqrt{\frac{a}{a-u}}\, du$$

(e) *(10 points)* Consider a geodesic curve in this space, described by the functions $u(s)$ and $\theta(s)$, where the parameter $s$ is chosen to be the arc length along the curve. Find the geodesic equation for $u(s)$, which should have the form

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[F(u,\theta)\,\frac{\mathrm{d}u}{\mathrm{d}s}\right] = \dots\,,$$

where $F(u,\theta)$ is a function that you will find. (Note that by writing $F$ as a function of $u$ and $\theta$, we are saying that it *could* depend on either or both of them, but we are not saying that it *necessarily* depends on them.) You need not simplify the left-hand side of the equation.

$$\frac{d}{d\lambda}\left[g_{ij}\frac{dx^j}{d\lambda}\right] = \frac{1}{2}\frac{\partial g_{k\ell}}{\partial x^i}\frac{dx^k}{d\lambda}\frac{dx^\ell}{d\lambda}$$

The geodesic equation for $u$ $\Rightarrow$ we want $j =$ "$u$"

$g_{ij}$ diagonal (no $d\theta du$ terms)

$\Rightarrow$ set $i =$ "$u$"

$$g_{uu} = \frac{a}{4u(a-u)}, \quad g_{\theta\theta} = u$$

Both nonzero $g_{kl}$ entries depend on $u$:

$$\frac{\partial g_{uu}}{\partial u} = -\frac{a}{4u^2(a-u)} + \frac{a}{4u(a-u)^2}$$

$$\frac{\partial g_{\theta\theta}}{\partial u} = 1 \qquad = \frac{2au-a^2}{4u^2(a-u)^2}.$$

$$\Rightarrow \frac{d}{ds}\left[\frac{a}{4u(a-u)}\frac{du}{ds}\right] = \frac{1}{2}\cdot\frac{2au-a^2}{4u^2(a-u)^2}\left(\frac{du}{ds}\right)^2 + \frac{1}{2}\left(\frac{d\theta}{ds}\right)^2$$

(f) *(8 points)* Similarly, find the geodesic equation for $\theta(s)$, which should have the form

$$\frac{\mathrm{d}}{\mathrm{d}s}\left[G(u,\theta)\,\frac{\mathrm{d}\theta}{\mathrm{d}s}\right] = \dots\,,$$

where $G(u,\theta)$ is a function that you will find. Again, you need not simplify the left-hand side of the equation.

For $\theta$:

$$\frac{d}{ds}\left[g_{ij}\frac{dx^j}{ds}\right] = \frac{1}{2}\frac{\partial g_{kl}}{\partial x^i}\frac{dx^k}{ds}\frac{dx^l}{ds}.$$

we want $j =$ "$\theta$", and $g$ diagonal means that we should set $i =$ "$\theta$"

Neither $g_{uu}$ nor $g_{\theta\theta}$ depend on $\theta$ $\Rightarrow$ the R.H.S. vanishes

$$\Rightarrow \frac{\partial}{\partial s}\left[\underbrace{u}\,\frac{d\theta}{ds}\right] = 0$$